Mathematical Induction-Day 1

**Context:** This lesson is designed for a high school Math Analysis course. It is the 3rd lesson of 7 in a unit encompassing also encompassing sequences, series, and the Binomial Theorem. This is the first of two lessons on induction.

**Objective:** Students will use mathematical induction to prove statements involving a positive integer \( n \) with 80 percent accuracy. These statements will include sums of sequences.

**SOL MA.6**
The student will use mathematical induction to prove formulas/statements.

**Material/Resources:** dominos

**Time Required:** 90 min

**Content and Instructional Strategies:**
1. Put up the following problems on the board for the students to work on as a warm up. Pick out students that have the correct answers to come to the board and write their answers. (20 min)
   a. Find the partial sum of the first 30 terms: 153 + 146 + 136 + 132 + ...
      i. \( a_n = -7n + 160; a_{30} = -210; S_{30} = 1545 \)
   b. Find the partial sum of the first 10 terms: 4 + .8 + .16 + .32 + ...
      i. \( r = .2; S_{10} = 5 \)
   c. Find the partial sum of the 4th through 11th terms given \( a_n = 3n + 7 \)
      i. \( S_{14} = 7 (19 + 58) = 539 \)
   d. Write in Sigma notation and find the partial sum: -2 + 8 -32 + ... + 2048
      i. \( \sum_{n=1}^{6} -2(-4)^{n-1} = -2 \left( \frac{1-(-4)^6}{1+4} \right) = 1638 \)
2. Answer any questions students have from the homework or last class. The homework will not be collected in order for students to use it to help study for their quiz on sequences and series given next class. (10 min)
3. Begin the new lesson by introducing the ladder problem. Ask students what they must know about the ladder in order for Romeo to make it to Juliet. Then, bring out the dominos to illustrate the logic behind induction. Ask the
students what they must know is true to be assured that all of the dominoes will fall down. (5 min)

4. Right the sequence 1, 3, 5, 7 on the board. Ask the students to use their knowledge from the last few lessons to determine the formula that will help predict more numbers in the sequence (answer is $a_n=2n-1$). Now ask them to form a conjecture about the sum of this sequence is in terms of n (answer is $n^2$). [Note: this is a different sum formula than what we would have arrived at from last class] Work some partial sums on the board to help explain how we get to this conclusion ($S_1=1=1^2; S_2=1+3=2^2; S_3=1+3+5=3^2$). Ask the students how they know that their answer is true. (5 min)

5. Bring up the peculiarity of the occurrence of prime numbers. Pierre de Fermat came up with an equation that he speculated output only prime numbers ($F_n=2^{2^n}+1$ for $n=0,1,...$). Work it out on the board for $n=0, 1, 2, 5$ (3, 5, 17, 4,294,967,297). Ask the students if all of these numbers are prime. Tell the students that after Fermat died, another mathematician, Leonhard Euler, found that $F_5=641*(6,700,417)$. So make the point that just because an equation works for a few numbers in the series, we don’t know for sure that it is true for all n. (5 min)

6. Ask the class if they have any suggestions on how to prove that the equation $S_n=n^2$ is true. Now bring the students attention to the set of dominoes set up at the front of the room. How would we be sure that all the dominoes will fall down? Step 1: We must be sure that the first domino will fall down when I push it. Step 2: we must be sure that if any nth domino falls down, the next domino (n+1) will also fall down. Illustrate how both of these conditions must be true. If Step 1 isn’t true, even if all of the other dominoes can fall down, (and the first doesn’t) none will. If step 2 fails, then at some point in the chain the dominoes will stop falling down, therefore they won’t all be true. Act this out before the class. Connect to the idea that a domino falling down is the same as $S_n$ being true for the nth domino. (5-7 min)

7. Tell the class that the algorithm just described is called mathematical induction. Let the students try to describe the necessary conditions and steps for $S_n$ and write them on the board. Walk the class through applying induction to prove that $S_n=n^2$. (7 min)

Step 1: $S_1 = 1 = 1^2$
Step 2: Assume true for $n=k$. Prove true for $k+1$. Assume that $S_k = 1 + 3 + 5 + 7 + \ldots + (2k-1) = k^2$. We must show that $S_{k+1} = (k + 1)^2$

\[
S_{k+1} = 1 + 3 + 5 + 7 + \cdots + (2k-1) + [2(k + 1) - 1] = S_k + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2
\]

Explain the Principle of Mathematical Induction (and relate this to previous problems) and write the necessary steps to follow on the board:

**Principle of Mathematical Induction:**

Let $P_n$ be a statement involving the positive integer $n$. If
1) $P_1$ is true, and
2) The truth of $P_k$ implies the truth of $P_{k+1}$ for every positive $k$

Then $P_n$ must be true for all positive integers $n$.

**Steps to follow Mathematical induction**

1) Show $P_k$ is true for $n = 1$ (or first term)
2) Assume true for $n = k$
3) Prove $P_{k+1}$ is true.
4) Write the conclusion.

8. Practice with the class finding $P_{k+1}$ given $P_k$. *(5-10 min)*

   a. $\frac{k^2(k+1)^2}{4} \rightarrow \frac{(k+1)^2(k+2)^2}{4}$ (note that for a formula like this, we simply substitute $k + 1$ in for $k$)

   b. $1+5+9+\ldots+[4(k-1)-3] + (4k-3) \rightarrow 1 + 5 + 9 + \cdots + (4k - 3) + (4k + 1)$ (note that this is a series, so we are adding on the $k + 1$ term in the series)

9. Do this example with the class. *(10 min)*

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S_n = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for all integers } n \geq 1
\]

   Step 1: When $n = 1$, $S_1 = \frac{1(1+1)(2+1)}{6} = \frac{1(2)(3)}{6}$

   Step 2: Now assume $S_k$ to be true,

   Step 3: We must show that $S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$

   a. $S_{k+1} = S_k + a_{k+1}$

   \[
   = \frac{k(k+1)(2k+1)}{6} + (k+1)^2
   \]
Step 4: If \( S_n \) is true for all positive integers \( n \).

10. Give the students the following equality to prove true using mathematical induction. While students are working, the teacher should go around answering students’ questions and assessing their understanding of the steps of induction. \( \text{(15 min)} \)

a. \( S_n = 2 + 4 + 6 + \cdots + 2n = n(n + 1) \)

Evaluation: The input that students give during the lecture and discussion will also help the teacher assess the students’ understanding of the new material. The teacher should also be walking around and gauging students’ progress on the equality given to the students to prove. Homework will be assigned \textit{next class}, after the material is completely covered, for another form of assessment.

Differentiation and Accommodations: To aid struggling students, the teacher will be available for individual help, especially during pairs work proving the last equality.

Ladder Problem:

Imagine a hero, Romeo, riding a horse towards a tall building (a castle). There is a ladder up the side of the building leading to the room where Juliet lives. The bottom step of the ladder is two meters or more (several feet or more) away from the ground. The ladder is not broken. It is in good condition. A person getting to each step of the ladder can climb to the next. Question: Can an able-bodied individual, Romeo, reach Juliet via the ladder? The answer is \textbf{yes} provided Romeo can get to the first or bottom-most step of the ladder. It is \textbf{no} otherwise. The main logic-related ideas in this brief story are as follows.

1. There is a long ladder to be climbed.
2. When any one step is reached, the next step can be reached. (The ladder must be in good condition for this to hold).
3. The first or bottom-most step can be reached.
This situation implies we (or Romeo) can reach each step of the ladder.

Note that the long ladder may have a finite number of steps, for example 183. Then we (or Romeo) can with enough time and patience, reach the last one, or any step in between.

On the other hand, we can imagine a ladder could have an infinite number of steps. For each step we take, a next is possible. For instance, the whole numbers we use for counting do not stop. Each whole number is followed by another - just add 1.

Now suppose or imagine we have a sequence of steps, a ladder, which goes on and on without stopping. Then with enough time and patience, we can reach anyone you mention. An example is met in counting. We can begin counting with the number 1, then 2, then 3 and so on.

When we begin to count, we may have only a finite number of objects to count. With a long enough life, and enough patience, the count will end. But if we count minutes there will always be one more to count. This minute count will never end. More precisely, each of us counters may end, but the counting of minutes in principle can continue. That is, this minute count can reach any large number you specify in advance with or without you. In principle all minutes after the beginning of the count will be met and counted.

To rephrase the above, on a ladder (or road) with finitely or infinitely many steps, the first step needs to be reachable. And from each step, the next step needs to be reachable. When this occurs, any whole number of steps along the road or ladder in question is reachable.

In practice, if each step takes time, the number of steps reachable will depend on how much time is available. Reach-ability here does not take into account the amount of time available, albeit people doing numerical computations on electronic computers must consider the latter.

CAUTION. The conclusion that all steps can be climbed or reached does not follow from the principle of mathematical induction if the ladder is broken, or if the first step is not reachable or if a tornado comes along, or if you break your ankle, etc. Check for these nasty situations when you want to use this principle to get a conclusion.

Found at: http://whyslopes.com/etc/ThreeSkillsForAlgebra/ch04.html